Perturbative analysis of finite stochastic networks

Stochastic synchronization, failure of mean-field theory and information processing analysis

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In this work, we develop a perturbative analysis of a finite neural network in terms of its internal noise and the initial conditions. We have proved that when the spectrum of the effective connectivity matrix has a negative eigenvalue with multiplicity 1, neurons can become perfectly correlated, invalidating the use of mean-field theories. Instead the linear approximation let us determine completely the information processing capabiliites of the network.

Introduction

• The mean-field theory developed by McKean, Vlasov, Sznitman, and others [1][2] is used to describe the activity of networks with infinitely many neurons, but it can't quantify finite size effects. The theory predicts that starting from independent initial conditions, neurons behave always independently during their time evolution (a phenomenon known as propagation of chaos).

• Accordingly, quantities like mutual information are always zero in networks with infinite neurons, so we can't determine the information processing capabilities of the system.

• We are in the realm of complexity, so what are the emergent properties of the system? Real systems are always noisy, so what is the role of random fluctuations in the cooperative interplay of the neurons? This in principle can be quantified in terms of stored, transmitted and modified information.

Perturbative analysis

Our network is described by the following rate equations:

$$dV_i(t) = \left[-\frac{1}{\tau}V_i(t) + \sum_{j=0}^{N-1} J_{ij} S(V_j(t)) + I_i(t)\right] dt + \sigma_B dB_i(t)$$

$$S(V_j(t)) = \frac{T_{MAX}}{1+e^{-\lambda(V_j - V_T)}}$$

with i=1,2,...,N. Here V_i and B_i are respectively the membrane potential and the Brownian noise of the i-th neuron, while J is the synaptic connectivity matrix and σ_B is the intensity of the noise. All the Brownians are supposed to be independent.

Now, using the following initial conditions:

$$\vec{V}(0) \sim \mathcal{N}(\vec{\mu}, \Sigma), \quad \vec{\mu} = \mu \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \Sigma = \sigma_0^2 \begin{bmatrix} 1 & C_0 & \cdots & C_0 \\ C_0 & 1 & \cdots & C_0 \\ \vdots & \vdots & \ddots & \vdots \\ C_0 & C_0 & \cdots & 1 \end{bmatrix}$$

replacing the following non-linear perturbative expansion:

$$V_i(t) \approx Y_0^i(t) + \sigma_0 Y_1^i(t) + \sigma_B Y_2^i(t) + \sigma_0^2 Y_3^i(t) + \sigma_B^2 Y_4^i(t) + \sigma_0 \sigma_B Y_5^i(t)$$

inside our system of eqs., and equating all the terms with the same order in σ_0 and σ_B, we obtain the expressions of Y_1, Y_2, ..., Y_5 as a function of Y_0 (the deterministic evolution). These can be used to compute the correlation between any pairs of neurons:

$$Corr(V_i(t), V_j(t)) = \frac{Cov(V_i(t), V_j(t))}{\sqrt{Var(V_i(t))\,Var(V_j(t))}}$$

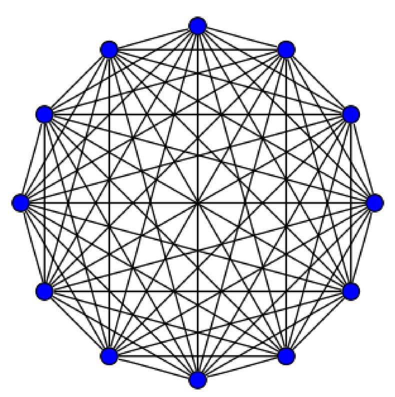
$$Cov(V_i(t), V_j(t)) = \sigma_0^2 \left[\sum_{k=0}^{N-1} \Phi_{ik}(t)\Phi_{jk}(t) + C_0 \sum_{\substack{k,l=0 \\ k \neq l}}^{N-1} \Phi_{ik}(t)\Phi_{jl}(t)\right] + \sigma_B^2 \sum_{k=0}^{N-1} \int_0^t \Phi_{ik}(t-s)\Phi_{jk}(t-s)\,ds$$

$$Var(V_i(t)) = Cov(V_i(t), V_i(t)), \quad \Phi(t) = e^{-\left(\frac{1}{\tau}I_d - W\right)t}, \quad W_{ij} = J_{ij} S'\left(Y_0^j(t)\right)$$

W is the effective connectivity matrix. It can be proved that when it has a negative eigenvalue with multiplicity 1, then:

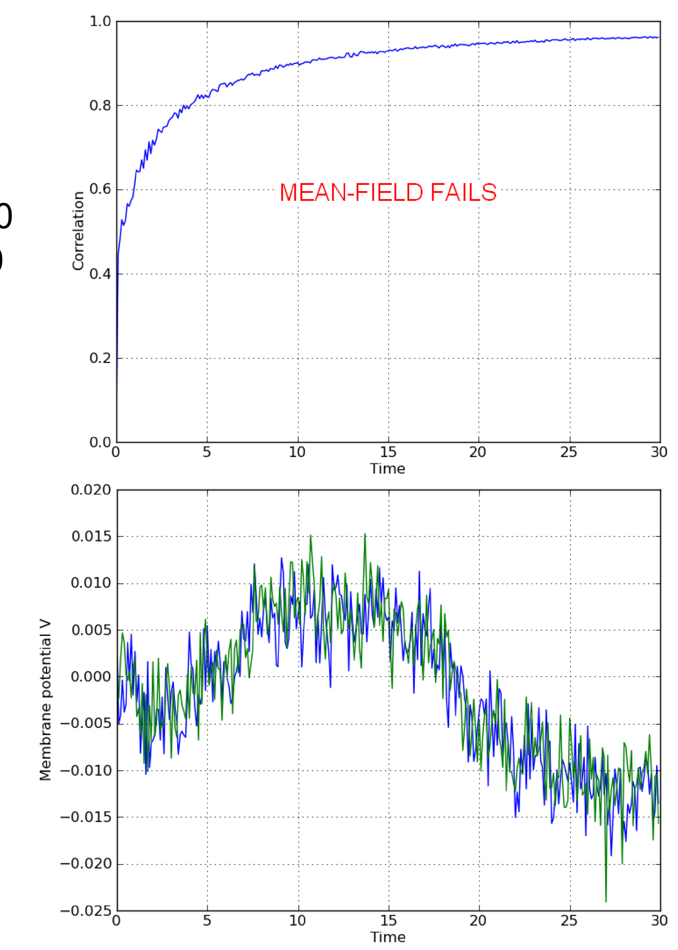
$$\lim_{t\to+\infty} Corr(V_i(t), V_j(t)) = 1$$

This phenomenon should be called stochastic synchronization, because the fluctuations of the noise become perfectly correlated. Here is an example obtained for a fully connected network:



N=12
Monte Carlo sim.s=2000
$\mu=0$, $\sigma_0=\sigma_B=0.01$, $C_0=0$
$T_{MAX}=1$, $\lambda=1$, $V_T=0$
$\tau=0.1$, $I=-20$

$$J = \frac{\overline{J}}{N-1} \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}, \quad \overline{J} = 40$$



The main idea of this analysis is to separate the deterministic evolution, i.e. the mean, of the system (numerically simulated) from its stochastic evolution (analytically calculated).

Explicit formulae for the correlation can be obtained also for circulant and block circulant matrices J. Using chemical graph theory [3] they can be obtained also for different types of symmetric connectivities, like ladders, the annulus, the torus, grids, cilinders, hypercubes, etc.

Information theory in the linear approximation

When σ_0 and σ_B are small enough, we can write:

$$V_i(t) \approx Y_0^i(t) + \sigma_0 Y_1^i(t) + \sigma_B Y_2^i(t)$$

Curiously, the formula for the correlation doesn´t change, but the probability density of the system becomes a multivariate normal distribution. Therefore its information theoretical quantities can be computed analytically. For example we have:

$$MI(X_1(t), X_2(t)) = \frac{1}{2}\log\left[\frac{det(\Sigma_{X_1}(t))\,det(\Sigma_{X_2}(t))}{det(\Sigma_{X_1 \otimes X_2}(t))}\right]$$

Mutual Information: it quantifies the degree of cooperation between 2 groups X_1 and X_2 of neurons

$$FI(X(t), I) = \left[\frac{\partial \mu(I,t)}{\partial I}\right]^T \Sigma^{-1}(I,t) \frac{\partial \mu(I,t)}{\partial I} + \frac{1}{2} Tr\left\{\left[\frac{\partial \Sigma(I,t)}{\partial I}\Sigma^{-1}(I,t)\right]^2\right\}$$

Fisher Information: it quantifies the capability of the random variable X (a collection of membrane potentials) to encode the input current I

where the mean μ(t) is known numerically and the covariance matrix Σ(t) is known from the linear perturbative analysis. Other quantities like the information storage, transfer and modification can be computed following the theory of cellular automata [4], after a time discretization of the system of our stochastic differential equations and taking the limit for $\Delta t \to 0$.

Discussion

• From this analysis it is evident that in general mean-field theories fail to describe networks of arbitrary (also infinite) size, because the phenomenon known as propagation of chaos does not always occur.

• Once the deterministic evolution of the system is known, we can compute explicitly the correlation structure of the system generated by its connectivity matrix. This allows us to reveal the hidden relation between the structural and functional connectivities of the system, a problem which is currently intensively investigated [5].

• This analysis sheds light on how information is processed in the network, as a function of its connectivity matrix. This is important from an engineering point of view to understand how the brain´s substrate works, even if the comprehension of higher cognitive functions could require other forms of information, like semantic information.

• In this work we have used only deterministically distributed synaptic weights, so the next step will be its extension to the case of randomly distributed weights. This is an important improvement toward the comprehension of the information processing power of biologically realistic networks with a small world connectivity matrix. In general this method requires the knowledge of all the eigenquantities of the connectivity matrix, that currently are known only for the Wigner matrix. This is the field of random matrix theory.

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