

Chaos propagation in a mean field theory of spiking neural networks

Independence, phase transitions and learning



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In this work, we demonstrate analytically and numerically the independence of spiking neurons when they are connected in large networks with Brownian noise, but without learning. We also explain why, at least intuitively, it's not possible to talk about independence and mean field theories in the case with learning. Finally, we give some hint about the study of phase transitions, in analogy with physics.

Introduction

- Eckel, Berens et al. [1] demonstrated experimentally that in the primary visual cortex of awake macaques correlation among neurons is small (of order from 10^{-3} to 10^{-2}).

- A simple and convincing theoretical explanation for this small correlation is not known yet.

- This phenomenon can be explained in a natural way by chaos propagation in the context of mean field theories, at least in the case **without learning**. This theory also demonstrates a stronger network effect: **independence**.

- Mean field theories also provide a framework for the study of **phase transitions**.

Why is this study important?

- Decorrelation reduces the Cramér-Rao bound, that defines the accuracy with which any unbiased decoding scheme can extract an estimate of an encoded quantity from the activity of a population of neurons. This demonstrates that **neural networks behave in an optimal way in terms of information coding**.

- This study provides a mean field equation that summarizes the activity of the whole network. Therefore we have a single equation, that is **much easier to study** (e.g. for bifurcation points) instead of the whole neural network system of stochastic differential equations.

- Extending this model in order to include the spatial dimension, we can describe extended neural tissues (e.g. the cerebral cortex), and therefore **improve the already existing neural field models** based on rate equations (e.g. the Liley model). This will lead us to analyse **macroscopic phase transitions** (e.g. anaesthetic induced conscious-unconscious transitions) from the point of view of well posed equations of spiking neurons, in order to include also information of spike times in the model.

These numerical simulations require the use of massive computational power (i.e. a CLUSTER) in order to improve the statistical precision with which correlation is computed.

Result n° 1: Analytical Proof

Stochastic FitzHugh-Nagumo model (the same work can be done with more biologically plausible models, like the Hodgkin-Huxley one):

$$\begin{cases} \frac{dV_i}{dt} = \left(V_i - \frac{V_i^3}{3} - W_i + I_d + (V_i - V_{rev}) \sum_{j \neq i} J_{ij} Y_j \right) \\ \frac{dW_i}{dt} = a(V_i + b - cW_i) \\ \frac{dY_i}{dt} = \alpha S(V_i)(1 - Y_i) - \beta Y_i + \sigma_Y(V_i, Y_i) \frac{dB_i^Y}{dt} \end{cases}$$

All the Brownians and the initial conditions are independent. No learning is used.

with $i=1,2,\dots,N$, and:

$$J_{ij} = \frac{1}{N} \left(\bar{J} + \sigma_J \frac{dB_i^J}{dt} \right), \quad S(V_i) = \frac{T_{max}}{1 + e^{-\lambda(V_i - V_T)}}$$

$$\sigma_Y(V_i, Y_i) = \sqrt{\alpha S(V_i)(1 - Y_i) + \beta Y_i} \chi(Y_i), \quad \chi(Y_i) = \begin{cases} \frac{\Lambda}{\Gamma e^{-\lambda(2Y_i - 1)}} & 0 < Y_i < 1 \\ 0 & Y_i \leq 0 \vee Y_i \geq 1 \end{cases}$$

Here we have shown the chemical synapse equation, but we can use the electrical one as well. Applying the theory developed by McKean, Vlasov, Sznitman, Tanaka and Villani [2][3] we obtain that, in the limit $N \rightarrow +\infty$, all the neurons behave like identical copies, that means they have the same law $p(t, V, W, Y)$, that affects their **MEAN FIELD EQUATION**:

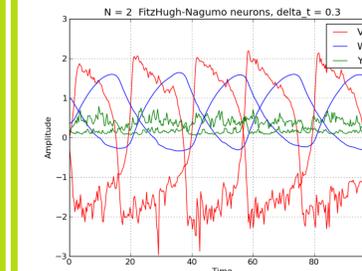
$$\begin{cases} \frac{dV_i}{dt} = V_i - \frac{V_i^3}{3} - W_i + I_d + \bar{J}(V_i - V_{rev}) \int_{\mathbb{R}^3} zp(t, x, y, z) dx dy dz + \sigma_J(V_i - V_{rev}) \left[\int_{\mathbb{R}^3} zp(t, x, y, z) dx dy dz \right] \frac{dB_i^J}{dt} \\ \frac{dW_i}{dt} = a(V_i + b - cW_i) \\ \frac{dY_i}{dt} = \alpha S(V_i)(1 - Y_i) - \beta Y_i + \sigma_Y(V_i, Y_i) \frac{dB_i^Y}{dt} \end{cases}$$

We have demonstrated that this stochastic mean field equation is well-posed, since it has a unique strong solution. This equation can also be equivalently described by its corresponding **FOKKER-PLANCK EQUATION**:

$$\begin{aligned} \frac{\partial p}{\partial t} = & \frac{1}{2} \sigma_J^2 \left[\int_{\mathbb{R}^3} zp(t, x, y, z) dx dy dz \right]^2 \frac{\partial^2}{\partial V^2} \left((V - V_{rev})^2 p \right) + \frac{1}{2} \frac{\partial^2}{\partial Y^2} \left(\sigma_Y^2(V_i, Y_i) p \right) + \\ & - \frac{\partial}{\partial V} \left[\left(V - \frac{V^3}{3} - W + I_d + \bar{J}(V - V_{rev}) \int_{\mathbb{R}^3} zp(t, x, y, z) dx dy dz \right) p \right] + \\ & - \frac{\partial}{\partial W} [a(V + b - cW)p] - \frac{\partial}{\partial Y} [(\alpha S(V)(1 - Y) - \beta Y)p] \end{aligned}$$

Result n° 2: Numerical Proof

Numerical algorithm used: explicit Milstein method (strong convergence order 1):



STOCHASTIC FITZHUGH-NAGUMO NEURAL NETWORK WITHOUT LEARNING

For the stochastic Hodgkin-Huxley model an implicit method is required, due to its stiffness.

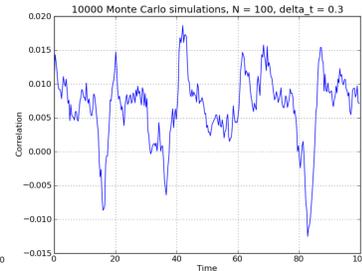
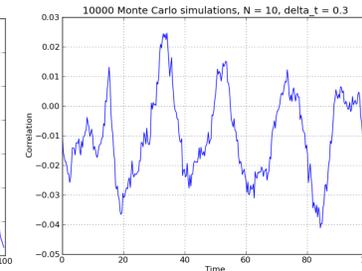
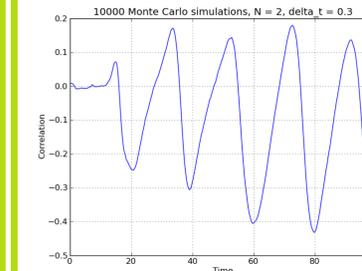
Parameters used:

$$\Delta t = 0.3, \bar{J} = 1, \sigma_J = 1, \Gamma = 0.3$$

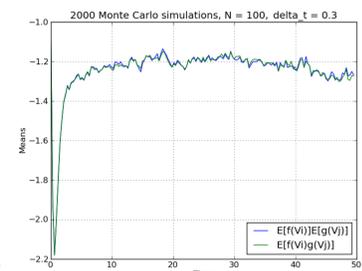
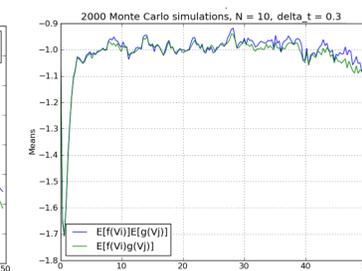
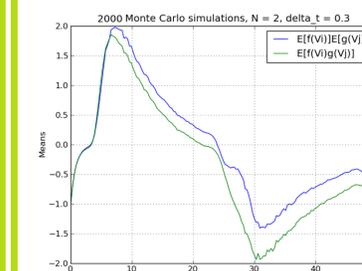
We have computed sample means using thousands of Monte Carlo simulations. The statistical error is:

$$Stat. Err. = \frac{1}{\sqrt{N^\circ \text{ Monte Carlo simulations}}}$$

Decorrelation increases with the number of neurons N...



... and also independence (factorization of the moments, here for example with $f(\cdot) = (\cdot)^2$ and $g(\cdot) = (\cdot)^3$):



Discussion

- We have demonstrated analytically and numerically that mean field theory provides a **natural explanation** for the experimentally observed [1] decorrelation of neurons in the primary visual cortex, at least in the case without learning.

- In the case with learning, instead, we know that STDP adjusts the connection strengths J_{ij} based on the relative timing of a particular neuron's output and input spikes. This means that the synaptic weights are changed according to a relation (the spiking time difference) between two neurons, and therefore this creates correlations. Thus, **at least intuitively, it shouldn't be possible to have strong decorrelation in the case with learning and to write a mean field theory for this system**.

- The McKean-Vlasov mean field equation of our model will let us to study more easily also mesoscopic phase transitions like the second order **phase transition from incoherence to synchronization** in the Kuramoto model. If our model is extended including the spatial dimension, it should be able to describe also macroscopic phase transitions, like the first order **phase transition from the conscious to the unconscious state** of the cerebral cortex caused by anaesthesia, and the **REM-SWS phase transition** during sleep, already described by the Liley rate model [4][5].

Bibliography

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